Faculty
Civil Engineering and GeoSciences

Remote Open book Exam	**CIE4190** **Slender Structures**
Total number of pages	**8 pages (excl cover)**
Date and time	**JAN-21-2022** from **13:30-16:30**
Responsible lecturer	**J.W. Welleman**

Only handwritten papers which have been uploaded (in pdf) in time will be assessed, Read also 'Additional Information'.

Exam questions (to be filled in by course examiner)

Total number of main topics with questions: **3** (each in separate block with time constraint)

☒ **questions may differ in weight** *(the time mentioned is an indicator for the weight)*

Use of tools and sources of information (to be filled in by course examiner)

Deliver your own work according to integrity statement:

- No collaboration or help "from others",
- No communication through any kind of medium,
- No distribution of files or answers to others,
- Etc.

Allowed:

☒ **books** ☒ **notes** ☒ **dictionaries** ☒ **syllabus**

☒ **formula sheets (see also below under 'additional information')** ☒ **calculators**

☒ **computer** ☐ **…**

☒ **scientific (graphical)calculator** ☒ **drawing material**

Additional information (if necessary to be filled in by the examiner)

- **Use your own examination paper with name and study number on all pages**
- **The question form contains fomula sheets which can be used.**
- **Papers will only be graded if the integrity statement has been uploaded.**
- **Upload in time, every part as ONE PDF file, with standard name format:**
  surname_student number_exam part example BOND_007_1

Exam graded by: (the marking period is 15 working days at most)

Every suspicion of fraud is reported to the Board of Examiners.

Mobile Phone OFF.

Problem 1: Elementary cases and MatrixMethod (50-60 min)

1.1 Defining elements from basic cases

Consider the cable element shown below. A constant horizontal force H is applied to both ends of the element. Under the assumption of arbitrary values for the positions of both nodes along the z-axis, the resulting vertical forces V at the ends of the elements can be related to the nodal positions (or displacements relative to an assumed initial position) in such a way that makes it possible to describe cables using the Matrix Method. For simplicity, we assume H is known and the cable does not elongate.

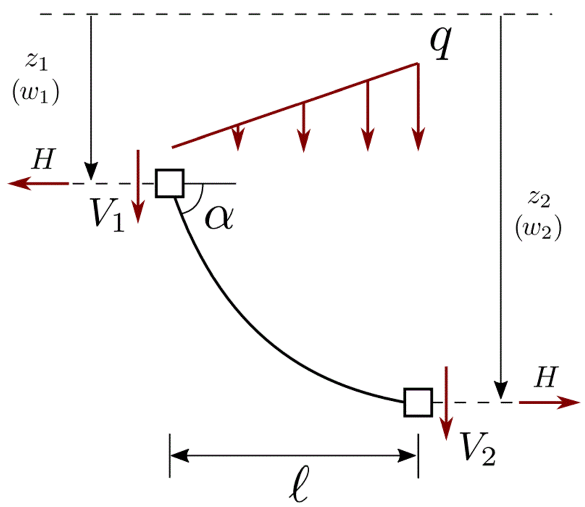

Figure 1: A cable element to be used within a larger structure through the Matrix Method.

Torsional problems can also be solved in a discrete way using the Matrix Method, provided that a suitable element definition is found. With that in mind, answer the following questions:

Questions:

a) Find the ODE that describes the position of this cable element and solve it in order to find its stiffness matrix $K^{(e)}$ and equivalent force vector $f^{(e)}$ such that $V^{(e)} = K^{(e)}z^{(e)} - f^{(e)}$. **Clearly show all intermediate steps of your derivation.**

b) What would change in your derivation if you were to define an element for a catenary cable? What degree of shape functions would be necessary when deriving $\boldsymbol{K}$ in that case? Clearly motivate your answer.

1.2 Matrix method system

Consider the following frame structure, to be modeled using the Matrix Method. Nodes are represented by squares and all connections between elements are to be considered rigid. Node and element numberings are given (element numbers are given between parentheses). The structure is fully clamped at Node 5 and supported both horizontally and vertically at Node 1. The additional horizontal support above Node 4 is not included in the discrete model, and is only present to ensure the shear beam cannot rotate. Element (1) deforms purely as a **cable**, Elements (2) and (3) include a **shear beam** with stiffness k and Element (4) deforms only in **extension** with stiffness EA. A constant horizontal load H is applied at the ends of the cable. Note that **the cable spans the whole length of the structure**. Between Nodes 2 and 4, the cable is perfectly bonded to the shear beam and therefore deforms together with it. Assume the cable does not deform axially and is long enough to fit the deformed shape of the frame.

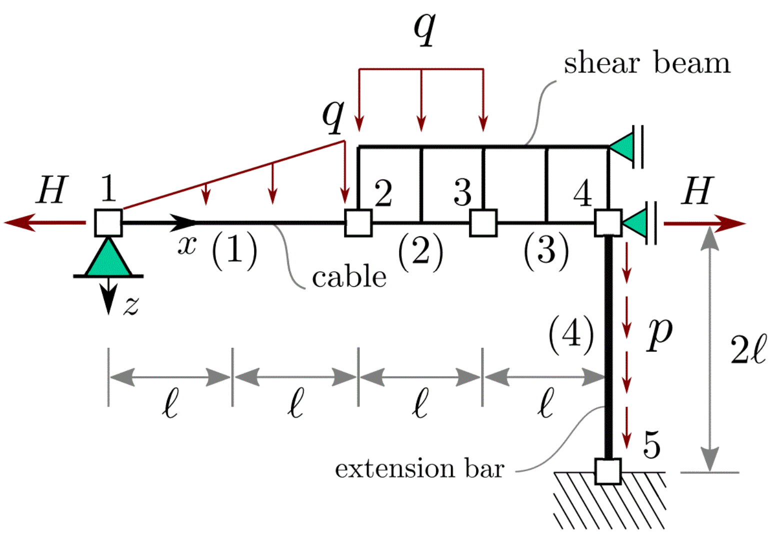

Figure 2: Frame structure with cable, extension and shear beam.

Given:

$k = 1000$ kN, $EA = 2000$ kN, $H = 500$ kN, $l = 2$ m, $q = 6$ kN/m, $p = 5$ kN/m

Questions:

a) List the DOFs associated with each element in the structure and identify the **element type** needed to describe each of the elements. With the given parameter values, compute the stiffness matrices and equivalent force vectors for Elements (1) and (4).

b) Using the method of your choice (e.g. ODE), obtain a **fully-parametric element definition** ($K^{(e)}, f^{(e)}$) to describe the deformation of Elements (2) and (3). Clearly motivate your choice of system and show all steps involved in your derivation (do not copy/paste your Maple script). Then, substitute the given parameter values and compute $K^{(2)}, f^{(2)}$ and $K^{(3)}$.

c) Assemble the global system of equations, apply boundary conditions and solve for the degrees of freedom.

d) Compute the sag at midspan of Element (1). Clearly show the steps taken in order to obtain it.

e) Instead of the approach you took in (**b**), would you be able to correctly model this structure by placing **overlapping elements** (one cable, one beam) between Nodes 2–3 and between Nodes 3–4? Why (not)? Would this strategy work in case the shear beam was replaced with an Euler-Bernoulli beam? Why (not)?

Problem 1: Elementary cases and MatrixMethod (50-60 min)

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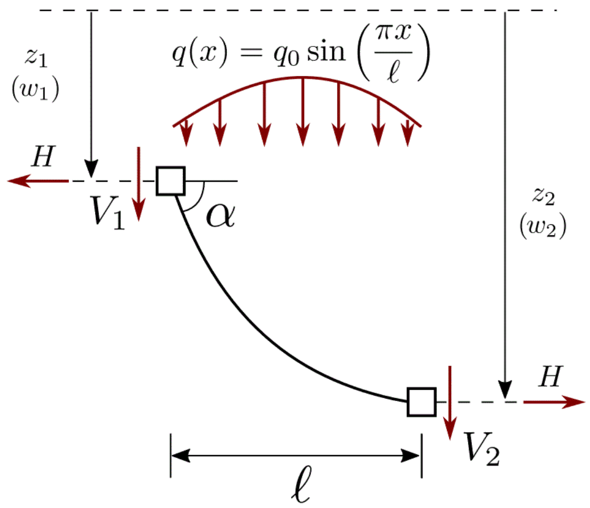

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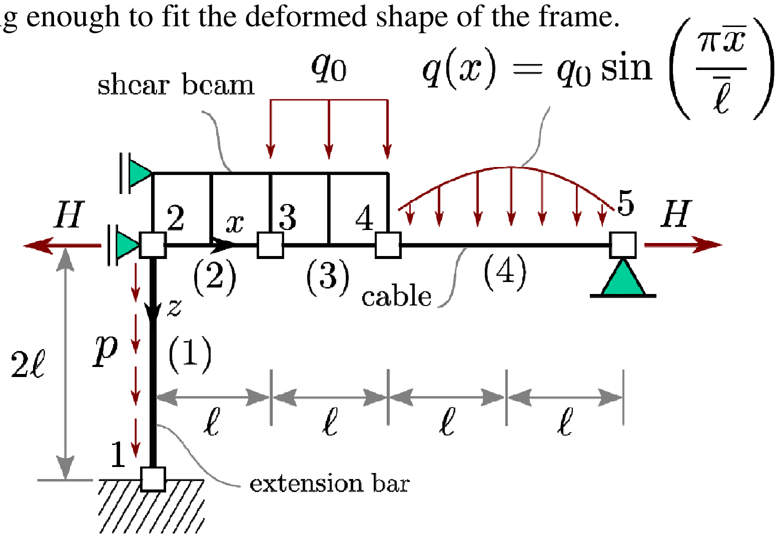

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Problem 1: Elementary cases and MatrixMethod (50-60 min)

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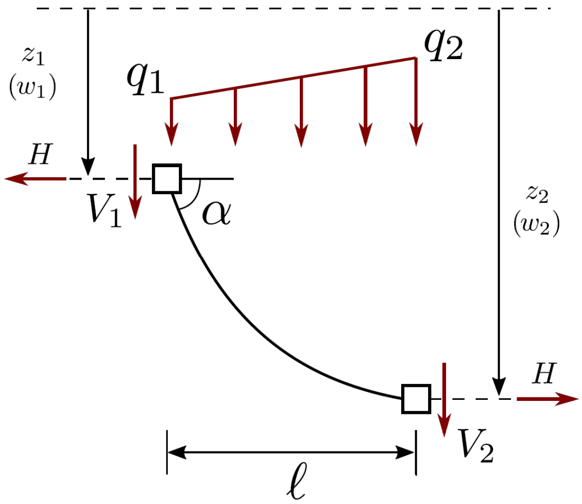

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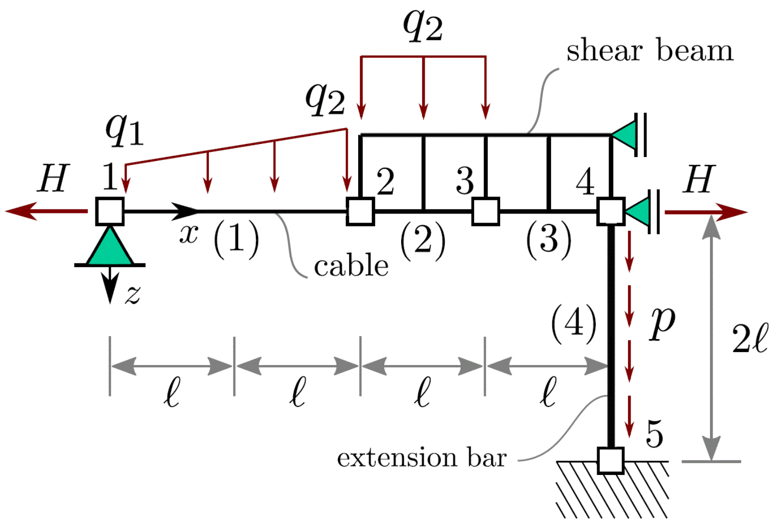

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Problem 2: Systems (50-60 min)

A beam in which deformation due to shear and due to bending is taken into account is shown in figure 3. The bending stiffness is denoted with EI and the effective shear stiffness is expressed with GA_{eff}. The beam has a rectangular cross section with width b and depth h and is made out of a linear elastic material with Young's modulus E and Poisson ratio ν.

The beam, with span l, is loaded with two loadcases A and B as indicated in fig. 3. This problem focusses on the influence of the shear deformation on the deflection of the beam.

NOTE : To simplify our computations we take Poisson's ratio fixed to 0.25.

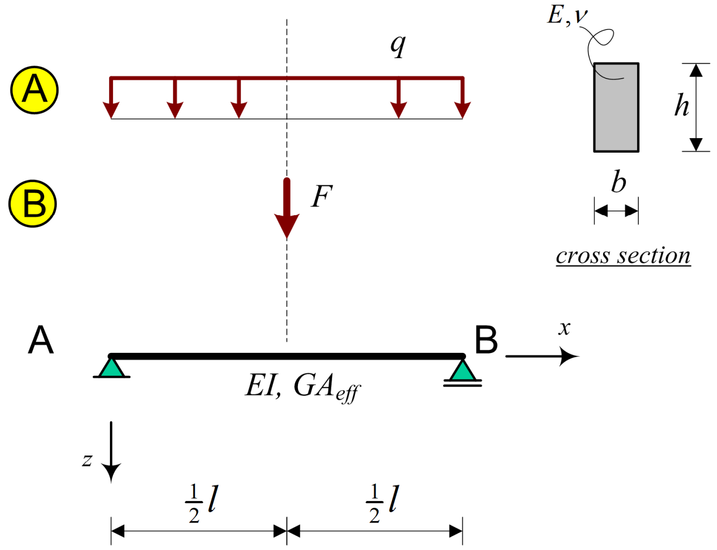

Figure 3: Beam with deformation due to both shear and bending.

Questions:

a) Specify the expressions for both the effective shear stiffness and bending stiffness in terms of the given parameters from fig. 3. Explain what is meant with "effective" in relation to the shear stiffness and do not forget to mention the units.

b) Respond to the following statement:
"The ratio between GA_{eff} and EI can be used to decide whether shear deformation has to be taken into account."

c) Setup the model needed to find the vertical displacement of the beam axis. Clearly show intermediate steps which leads to your model and also clearly show all unknowns in your model and the required conditions to solve these unknowns.

d) Find an expression for the deflection at midspan for both loadcases A and B and clearly specify the influence of the shear deformation in these expressions. Relate these expressions to the standard deflection without shear deformation.

e) Derive a design criterion based on slenderness, which limits the deflection at midspan due to shear deformation to 5%.

f) Will your design criterion be affected by the static system used (static determinate or static indeterminate) and if so in what way? Motivate your answer. (qualitative answer required)

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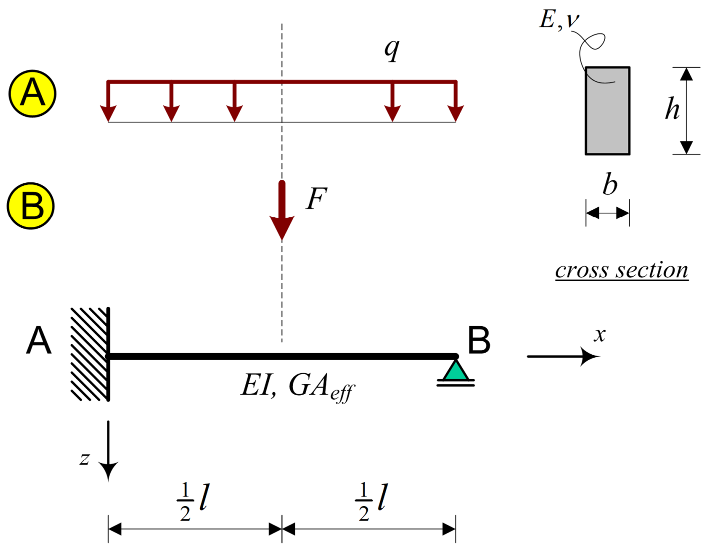

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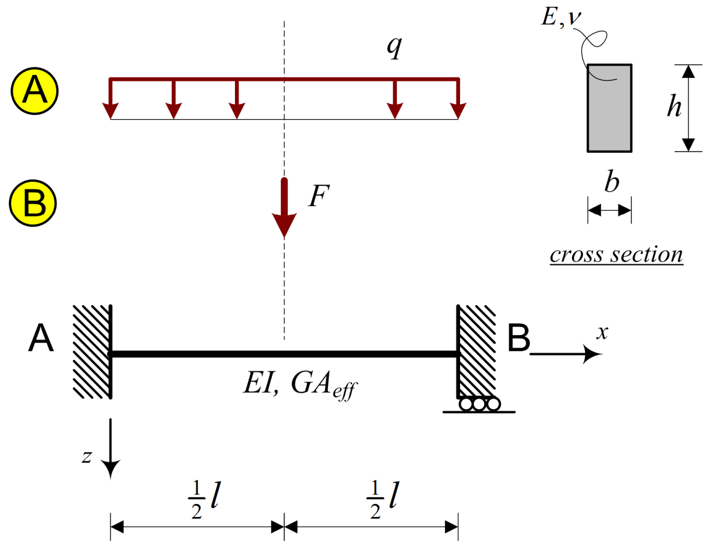

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Problem 3: Beams and Arches (50-60 min)

For a pedestrian bridge with a span of 20 m the designer came up with a "wild" idea. Not a straight beam but an eye catcher for the town with main girder constructed as an arch which will be clamped at both banks of the town river. The shape of the arch has multiple points of inflection as can be seen from fig. 4. The geometry of the arch axis is described with the following formula:

$$z(x) = \frac{-300x^2(l-x)^2}{10^7} \qquad [\mathrm{m}]$$

Within the design team there is some debate about the performance of this design compared to a much cheaper straight beam. This is the focus of this problem and for that only one specific load case is considered with a distributed load q on half of the span as shown in fig. 4. The bending stiffness EI is specified and axial deformation as well as shear deformation are assumed to be very small.

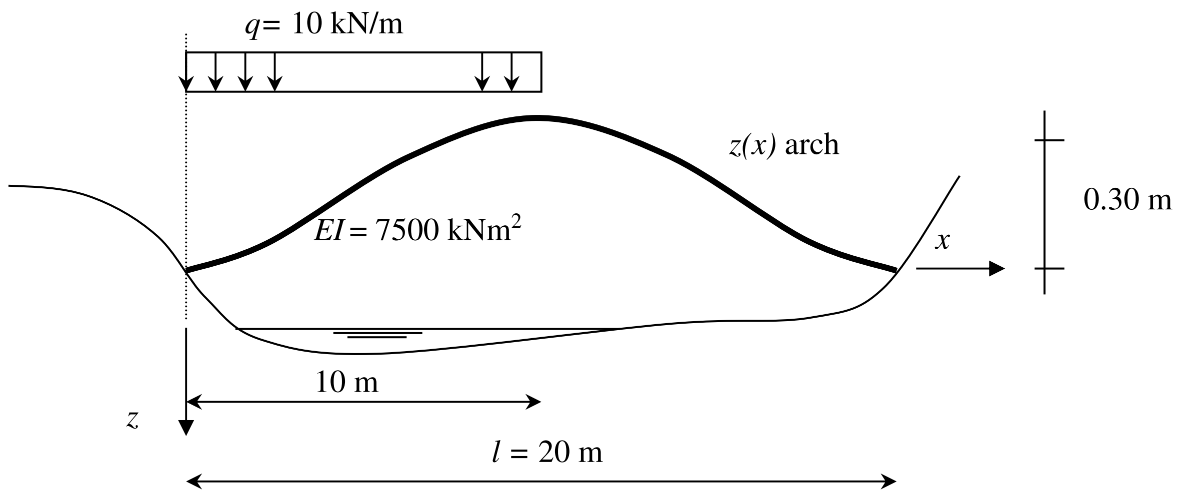

Figure 4: Arch structure

Questions:

a) Describe in a few lines the load carrying capacity of this "wild design" and how you can find both the force distribution and deflection. Explain relevant parameters and definition of signs in your answer, show the unknowns and clearly specify the equations needed to solve the unknowns.

b) Apply your method for the given loadcase and specify all support reactions with the actual directions, acting on the structure. Clearly show how you obtained the values and directions by specifying some intermediate steps. (do not copy your MAPLE script!)

c) Draw the moment distribution (along the x-axis, so not the axis of the arch!). Specify the sign or deformation symbols in your graph and the values at characteristic points. Also include in this diagram the moment distribution for the straight beam.

d) In order to judge the performance of this design add data which you think is also relevant to consider and assess this. Elaborate on the pros and cons of this design compared to a straight beam. You also may give some advice on design changes to improve the performance.

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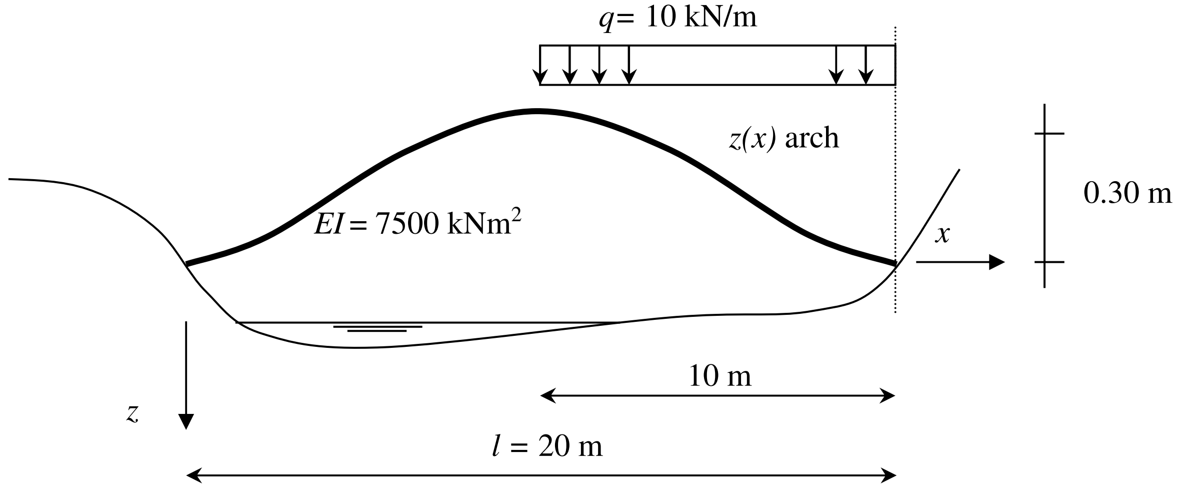

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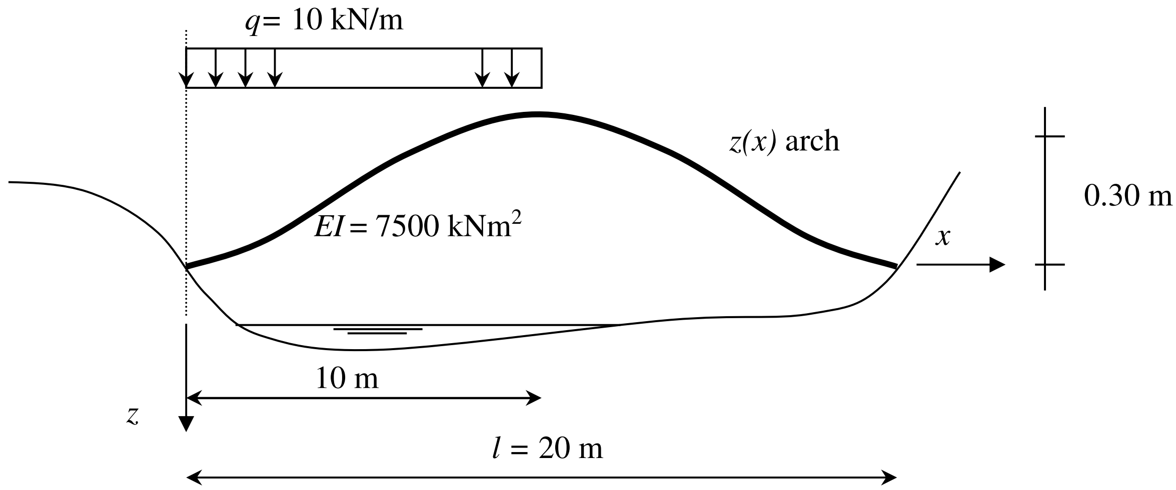
$$z(x) = \frac{-300x^2(l-x)^2}{10^7} \qquad [\text{m}]$$

Within the design team there is some debate about the performance of this design compared to a cheaper parabolic, clamped arch with the same maximum height of 0.3 m. This is the focus of this problem and for that only one specific load case is considered with a distributed load q on half of the span as shown in fig. 4. The bending stiffness EI is specified and axial deformation as well as shear deformation are assumed to be very small.



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d) In order to judge the performance of this design add data which you think is also relevant to consider and assess this. Elaborate on the pros and cons of this design compared to a parabolic arch. You also may give some advice on design changes to improve the performance.

FORMULAS

Temperature:

$$N = EA(\varepsilon - \varepsilon^T); \qquad M = EI(\kappa - \kappa^T)$$

$$\varepsilon^T = \frac{\alpha}{A}\int_A T(x,z)\mathrm{d}A; \qquad \kappa^T = \frac{\alpha}{I}\int_A zT(x,z)\mathrm{d}A$$

Stress distributions in beams:

$$\sigma(z) = \frac{N}{A} + \frac{Mz}{I} \qquad G = \frac{E}{2(1+\nu)}\ [\mathrm{N/mm^2}]$$

$$\tau(z) = \frac{6V\left(\frac{1}{4}h^2 - z^2\right)}{bh^3}$$ rectangular crosssections

Arch:

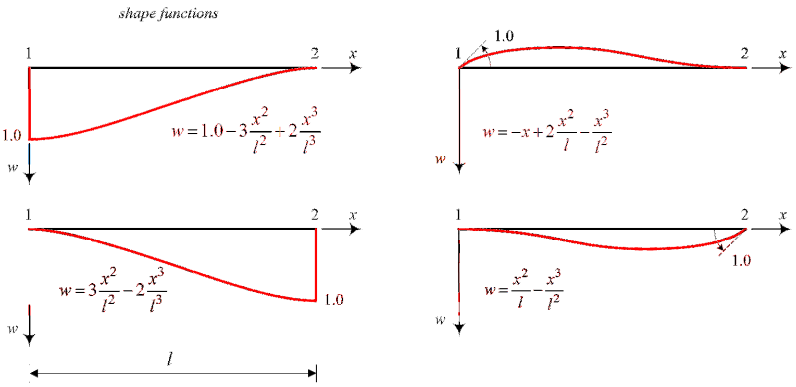
$$H = -\frac{\int_{arch} \frac{M^a z}{EI}\mathrm{d}x}{\int_{arch} \frac{z^2}{EI}\mathrm{d}x + \frac{l}{EA}}$$

Cable:

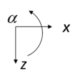
$$H^2 = \frac{q^2 l^3}{24\Delta};$$

$$f = \sqrt{\frac{3}{8} l\,\Delta} \text{ or } \Delta = \frac{8f^2}{3l}$$

Matrix Methods



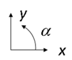
$$\begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} & 0 & \frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} & 0 & \frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}$$



$$R = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$\bar{f} = R\ f \quad \text{and} \quad f = R^T \bar{f}$$

$$\bar{u} = R\ u \quad \text{and} \quad u = R^T \bar{u}$$



$$R = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

Math tools:

$$y = Ae^{-\beta x}\sin(\beta x + \omega)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\sqrt{2}\,\beta A e^{-\beta x}\sin\left(\beta x + \omega - \tfrac{1}{4}\pi\right)$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 2\beta^2 A e^{-\beta x}\sin\left(\beta x + \omega - \tfrac{1}{2}\pi\right)$$

$$\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} = -2\sqrt{2}\beta^3 A e^{-\beta x}\sin\left(\beta x + \omega - \tfrac{3}{4}\pi\right)$$

$$\int \sqrt{1 + x^2}\,\mathrm{d}x \approx \int \left[1 + \frac{1}{2}\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2\right]\mathrm{d}x$$

$$\int_0^a \sin^2\frac{\pi x}{a}\mathrm{d}x = \frac{a}{2} \qquad \int_0^a x\sin\frac{\pi x}{a}\mathrm{d}x = \frac{a^2}{\pi}$$

$$\int_0^a x^2\sin\frac{\pi x}{a}\mathrm{d}x = \frac{a^3(\pi^2 - 4)}{\pi^3}$$

catenary solution:

$$z = -\frac{H}{q}\cosh\left(-\frac{qx}{H} + C_1\right) + C_2$$

basic math:

$$e^{\ln(a)} = a; \quad e^{-\ln(a)} = \tfrac{1}{a}; \quad z(x) = \frac{4fx(l-x)}{l^2}\ (parabola)$$

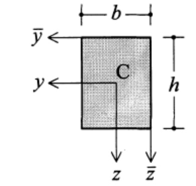
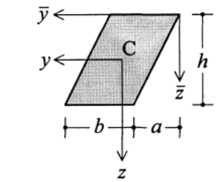
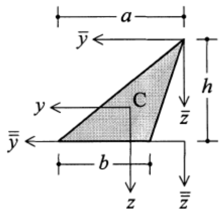
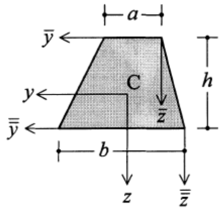
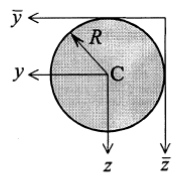
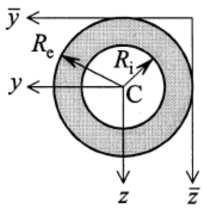
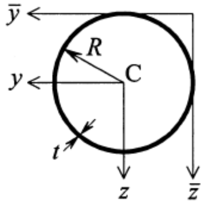
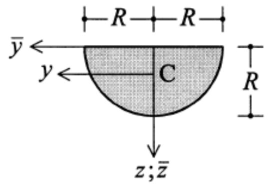
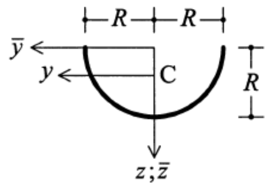
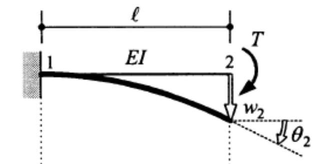
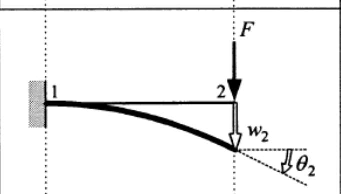
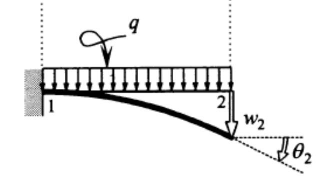
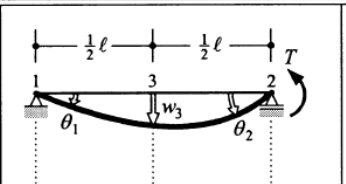
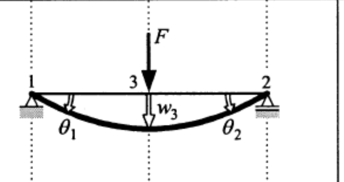
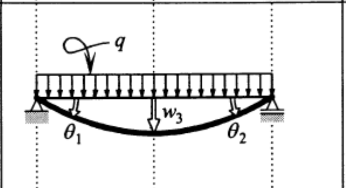
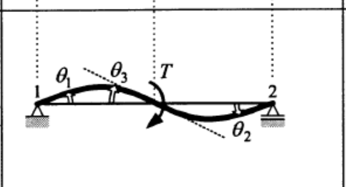
Figure	Area, coordinates centroid C	Second moments of area	
		centroidal	other
	Rectangle $A = bh$ $\bar{y}_C = \frac{1}{2}b$ $\bar{z}_C = \frac{1}{2}h$	$I_{yy} = \frac{1}{12}b^3h$ $I_{zz} = \frac{1}{12}bh^3$ $I_{yz} = 0$	$I_{\bar{y}\bar{y}} = \frac{1}{3}b^3h$ $I_{\bar{z}\bar{z}} = \frac{1}{3}bh^3$ $I_{\bar{y}\bar{z}} = \frac{1}{4}b^2h^2$
	Parallelogram $A = bh$ $\bar{y}_C = \frac{1}{2}(a+b)$ $\bar{z}_C = \frac{1}{2}h$	$I_{yy} = \frac{1}{12}(a^2+b^2)bh$ $I_{zz} = \frac{1}{12}bh^3$ $I_{yz} = \frac{1}{12}abh^2$	$I_{\bar{z}\bar{z}} = \frac{1}{3}bh^3$
	Triangle $A = \frac{1}{2}bh$ $\bar{y}_C = \frac{1}{3}(2a-b)$ $\bar{z}_C = \frac{2}{3}h$	$I_{yy} = \frac{1}{36}(a^2-ab+b^2)bh$ $I_{zz} = \frac{1}{36}bh^3$ $I_{yz} = \frac{1}{72}(2a-b)bh^2$	$I_{\bar{z}\bar{z}} = \frac{1}{4}bh^3$ $I_{\bar{y}\bar{z}} = \frac{1}{8}(2a-b)bh^2$ $I_{\bar{\bar{z}}\bar{\bar{z}}} = \frac{1}{12}bh^3$
	Trapezium $A = \frac{1}{2}(a+b)h$ $\bar{z}_C = \frac{1}{3}\frac{a+2b}{a+b}h$	$I_{zz} = \frac{1}{36}\frac{a^2+4ab+b^2}{a+b}h^3$	$I_{\bar{z}\bar{z}} = \frac{1}{12}(a+3b)h^3$ $I_{\bar{\bar{z}}\bar{\bar{z}}} = \frac{1}{12}(3a+b)h^3$
	Circle $A = \pi R^2$	$I_{yy} = I_{zz} = \frac{1}{4}\pi R^4$ $I_{yz} = 0$ $I_p = \frac{1}{2}\pi R^4$	$I_{\bar{y}\bar{y}} = I_{\bar{z}\bar{z}} = \frac{5}{4}\pi R^4$ $I_{\bar{y}\bar{z}} = \pi R^4$

Figure	Area, coordinates centroid C	Second moments of area	
		centroidal	other
	Thick-walled ring $A = \pi(R_e^2 - R_i^2)$	$I_{yy} = I_{zz} = \frac{1}{4}\pi(R_e^4 - R_i^4)$ $I_{yz} = 0$ $I_p = \frac{1}{2}\pi(R_e^4 - R_i^4)$	
	Thin-walled ring $A = 2\pi Rt$	$I_{yy} = I_{zz} = \pi R^3 t$ $I_{yz} = 0$ $I_p = 2\pi R^3 t$	$I_{\bar{y}\bar{y}} = I_{\bar{z}\bar{z}} = 3\pi R^3 t$
	Semicircle $A = \frac{1}{2}\pi R^2$ $\bar{y}_C = 0$ $\bar{z}_C = \frac{4}{3\pi}R$ $= 0.424R$	$I_{yy} = \frac{1}{8}\pi R^4 = 0.393R^4$ $I_{zz} = (\frac{\pi}{8} - \frac{8}{9\pi})R^4$ $= 0.110R^4$ $I_{yz} = 0$	$I_{\bar{y}\bar{y}} = I_{\bar{z}\bar{z}} = \frac{1}{8}\pi R^4$ $I_{\bar{y}\bar{z}} = 0$
	Semicircular ring $A = \pi Rt$ $\bar{y}_C = 0$ $\bar{z}_C = \frac{2}{\pi}R$ $= 0.637R$	$I_{yy} = \frac{1}{2}\pi R^3 t$ $I_{zz} = (\frac{\pi}{2} - \frac{4}{\pi})R^3 t$ $= 0.298R^3 t$ $I_{yz} = 0$	$I_{\bar{y}\bar{y}} = I_{\bar{z}\bar{z}} = \frac{1}{2}\pi R^3 t$ $I_{\bar{y}\bar{z}} = 0$

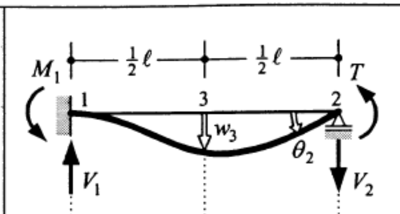
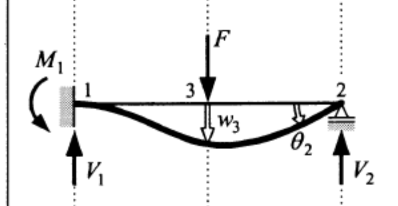
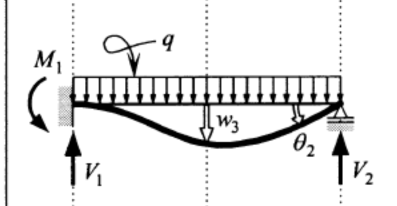
forget-me-nots

(1)		$\theta_2 = \dfrac{T\ell}{EI}$; $w_2 = \dfrac{T\ell^2}{2EI}$
(2)		$\theta_2 = \dfrac{F\ell^2}{2EI}$; $w_2 = \dfrac{F\ell^3}{3EI}$
(3)		$\theta_2 = \dfrac{q\ell^3}{6EI}$; $w_2 = \dfrac{q\ell^4}{8EI}$

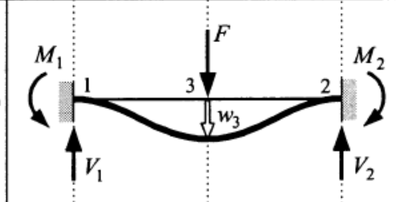
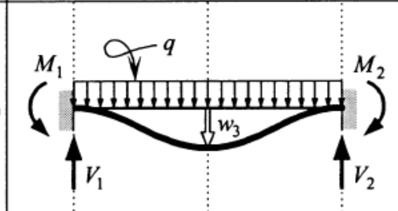
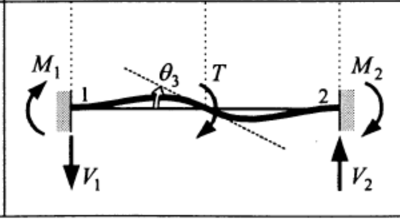
simply supported beam (statically determinate)

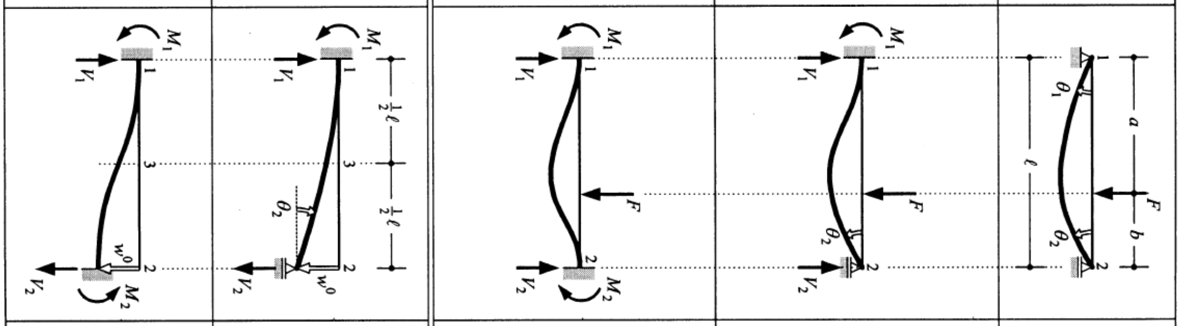
(4)		$\theta_1 = \dfrac{1}{6}\dfrac{T\ell}{EI}$; $\theta_2 = \dfrac{1}{3}\dfrac{T\ell}{EI}$; $w_3 = \dfrac{1}{16}\dfrac{T\ell^2}{EI}$
(5)		$\theta_1 = \theta_2 = \dfrac{1}{16}\dfrac{F\ell^2}{EI}$; $w_3 = \dfrac{1}{48}\dfrac{F\ell^3}{EI}$
(6)		$\theta_1 = \theta_2 = \dfrac{1}{24}\dfrac{q\ell^3}{EI}$; $w_3 = \dfrac{5}{384}\dfrac{q\ell^4}{EI}$
(a)		$\theta_1 = \theta_2 = \dfrac{1}{24}\dfrac{T\ell}{EI}$; $\theta_3 = \dfrac{1}{12}\dfrac{T\ell}{EI}$; $w_3 = 0$

statically indeterminate beam (one fixed end)

(7)		$\theta_2 = \dfrac{1}{4}\dfrac{T\ell}{EI}$; $w_3 = \dfrac{1}{32}\dfrac{T\ell^2}{EI}$ $M_1 = \dfrac{1}{2}T$; $V_1 = V_2 = \dfrac{3}{2}\dfrac{T}{\ell}$
(8)		$\theta_2 = \dfrac{1}{32}\dfrac{F\ell^2}{EI}$; $w_3 = \dfrac{7}{768}\dfrac{F\ell^3}{EI}$ $M_1 = \dfrac{3}{16}F\ell$; $V_1 = \dfrac{11}{16}F$; $V_2 = \dfrac{5}{16}F$
(9)		$\theta_2 = \dfrac{1}{48}\dfrac{q\ell^3}{EI}$; $w_3 = \dfrac{1}{192}\dfrac{q\ell^4}{EI}$ $M_1 = \dfrac{1}{8}q\ell^2$; $V_1 = \dfrac{5}{8}q\ell$; $V_2 = \dfrac{3}{8}q\ell$

statically indeterminate beam (two fixed ends)

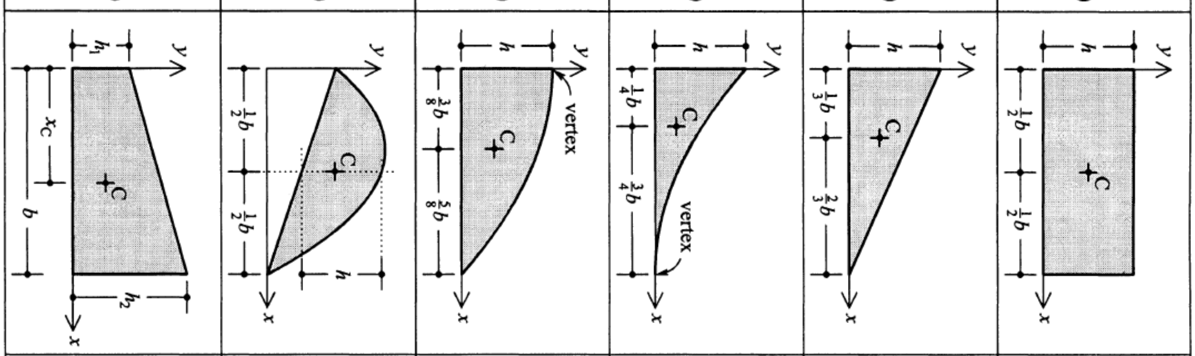
(10)		$w_3 = \dfrac{1}{192}\dfrac{F\ell^3}{EI}$ $M_1 = M_2 = \dfrac{1}{8}F\ell$; $V_1 = V_2 = \dfrac{1}{2}F$
(11)		$w_3 = \dfrac{1}{384}\dfrac{q\ell^4}{EI}$ $M_1 = M_2 = \dfrac{1}{12}q\ell^2$; $V_1 = V_2 = \dfrac{1}{2}q\ell$
(b)		$\theta_3 = \dfrac{1}{16}\dfrac{T\ell}{EI}$; $w_3 = 0$ $M_1 = M_2 = \dfrac{1}{4}T$; $V_1 = V_2 = \dfrac{3}{2}\dfrac{T}{\ell}$



(c)	$\theta_1 = \frac{Fab(\ell+b)}{6EI\ell} = \frac{F\ell^2}{6EI}\left(2\frac{a}{\ell} - 3\frac{a^2}{\ell^2} + \frac{a^3}{\ell^3}\right)$ $\theta_2 = \frac{Fab(\ell+a)}{6EI\ell} = \frac{F\ell^2}{6EI}\left(\frac{a}{\ell} - \frac{a^3}{\ell^3}\right)$
(d)	$M_1 = \frac{Fb(\ell^2-b^2)}{2\ell^2} = F\ell\left(\frac{a}{\ell} - \frac{3}{2}\frac{a^2}{\ell^2} + \frac{1}{2}\frac{a^3}{\ell^3}\right)$ $V_1 = \frac{Fb(3\ell^2-b^2)}{2\ell^3} = F\left(1 - \frac{3}{2}\frac{a^2}{\ell^2} + \frac{1}{2}\frac{a^3}{\ell^3}\right)$ $V_2 = \frac{Fa^2(3\ell-a)}{2\ell^3} = F\left(\frac{3}{2}\frac{a^2}{\ell^2} - \frac{1}{2}\frac{a^3}{\ell^3}\right)$ $\theta_2 = \frac{Fa^2b}{4EI\ell} = \frac{F\ell^2}{4EI}\left(\frac{a^2}{\ell^2} - \frac{a^3}{\ell^3}\right)$
(e)	$M_1 = \frac{Fab^2}{\ell^2} = F\ell\left(\frac{a}{\ell} - 2\frac{a^2}{\ell^2} + \frac{a^3}{\ell^3}\right)$ $V_1 = \frac{Fb^2(\ell+2a)}{\ell^3} = F\left(1 - 3\frac{a^2}{\ell^2} + 2\frac{a^3}{\ell^3}\right)$ $M_2 = \frac{Fa^2b}{\ell^2} = F\ell\left(\frac{a^2}{\ell^2} - \frac{a^3}{\ell^3}\right)$ $V_2 = \frac{Fa^2(\ell+2b)}{\ell^3} = F\ell\left(3\frac{a^2}{\ell^2} - 2\frac{a^3}{\ell^3}\right)$
(f)	$M_1 = \frac{3EI}{\ell^2}w^0;\quad V_1 = V_2 = \frac{3EI}{\ell^3}w^0$ $\theta_2 = \frac{3}{2}\frac{w^0}{\ell}$ $\theta_3 = \frac{9}{8}\frac{w^0}{\ell};\quad w_3 = \frac{5}{16}w^0$
(g)	$M_1 = M_2 = \frac{6EI}{\ell^2}w^0;\quad V_1 = V_2 = \frac{12EI}{\ell^3}w^0$ $\theta_3 = \frac{3}{2}\frac{w^0}{\ell};\quad w_3 = \frac{1}{2}w^0$

settlements (f), (g) — *support reactions and rotations at the beam ends* (c), (d), (e)

properties of plane figures to be used for the moment-area theorems



(1)	rectangle: $y = h$ $A = bh$ $x_C = \frac{1}{2}b$
(2)	triangle: $y = h\left\{1 - \frac{x}{b}\right\}$ $A = \frac{1}{2}bh$ $x_C = \frac{1}{3}b$
(3)	parabola: $y = h\left\{1 - \frac{x}{b}\right\}^2$ $A = \frac{1}{3}bh$ $x_C = \frac{1}{4}b$
(4)	parabola: $y = h\left\{1 - \left(\frac{x}{b}\right)^2\right\}$ $A = \frac{2}{3}bh$ $x_C = \frac{3}{8}b$
(5)	parabola: $A = \frac{2}{3}bh$ $x_C = \frac{1}{2}b$
(6)	trapezium: $y = h_1 + (h_2 - h_1)\frac{x}{b}$ $A = \frac{1}{2}b(h_1 + h_2)$ $x_C = \frac{1}{3}b\frac{h_1 + 2h_2}{h_1 + h_2}$

Timoshenko − beam

$$\frac{d^2 w(x)}{dx^2} = -\frac{q(x)}{GA_{eff}} - \frac{M(x)}{EI}$$

$$EI\frac{d^2\varphi(x)}{dx^2} - GA_{eff}\left(\frac{dw(x)}{dx} + \varphi(x)\right) = 0 \quad (1)$$

$$GA_{eff}\left(\frac{d^2 w(x)}{dx^2} + \frac{d\varphi(x)}{dx}\right) = -q(x) \quad (2)$$